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Generalizations of Continuity

By BUCHANAN CARGAL

Let X be a Hausdorff space in which each neighborhood is an inexhaustible set (a set of Baire's second category), and let $\{N\}$ denote the system of neighborhoods in X . Thus $N(x)$ is a neighborhood of the element x . Let Y be a regular separable Hausdorff space and let $\{M\}$ denote the system of neighborhoods in Y . If f is a function on X into Y and ξ is a point of X , then f is continuous at ξ if for every $M(f(\xi))$ and for every $N(\xi)$, the set $N(\xi)E[x:f(x) \in M(f(\xi))]$ contains ξ as an interior point. A direct method for generalizing continuity is to weaken the requirements on the set $N(\xi)E[x:f(x) \in M(f(\xi))]$. For example, if $N(\xi)E[x:f(x) \in M(f(\xi))]$ contains an N (not necessarily an $N(\xi)$), then f is neighborly at ξ [1]¹. Other generalizations are obtained by requiring that $N(\xi)E[x:f(x) \in M(f(\xi))]$ contains an inexhaustible set, a set residual on a neighborhood (that is, a set whose complement on the neighborhood is exhaustible, or of Baire's first category), a non-denumerable set, or a non-null dense-in-itself subset. The purpose of this paper is to unify the study of generalized continuity by developing a concept which includes each of the generalizations mentioned above as special cases. It should be observed that properties of $N(\xi)E[x:f(x) \in M(f(\xi))]$ do not characterize functions which are semi-continuous or cliquish [2] at ξ .

DEFINITION I a. Let λ be a set property defined for subsets of X and let S be a subset of X . The point ξ is said to be a *point of λ approach by S* if for every $N(\xi)$, $N(\xi)S$ has property λ .²

b. The point ξ is said to be a *point of $k\lambda$ approach (concentrated λ approach) by S* if there exists an $N(\xi)$ such that for every N contained in $N(\xi)$, NS has property λ .

LEMMA I. For any property λ and for any set S in X , the set E of points of λ approach by S which are not points of $k\lambda$ approach by S is nowhere dense.

PROOF. If a neighborhood N contains a point of E , then there exists a neighborhood N_1 contained in N such that N_1S does not have property λ . Thus no element of N_1 belongs to E .

DEFINITION II a. Let f be a function on X into Y . The point

¹Numbers in brackets refer to the list of references.

²Definitions I and II are essentially those introduced by Blumberg [3] for the case when the property λ is inexhaustibility

ξ is said to be a *point of λ f -approach by S* if for every $M(f(\xi))$ and for every $N(\xi)$, $N(\xi)S \cap E[x:f(x) \in M(f(\xi))]$ has property λ .

b. The point ξ is said to be a *point of $k\lambda$ (concentrated λ) f -approach by S* if for every $M(f(\xi))$, there exists an $N(\xi)$ such that for every N contained in $N(\xi)$, $NSE[x:f(x) \in M(f(\xi))]$ has property λ .

DEFINITION III. The property λ is said to be an ascending set property if for every set S which has property λ and for every subset A of X , it is true that $(S+A)$ has property λ .

LEMMA II. Let λ be an ascending set property, let $f(x)$ be defined on X and let S be a subset of X . The set E of points of λ f -approach by S which are not points of $k\lambda$ f -approach by S is an exhaustible set.

PROOF. Let $\{M^r\}$ be a countable set of neighborhoods which are equivalent to $\{M\}$ and choose M_j^r from $\{M^r\}$. By Lemma I, the set E_j of points of λ approach by $SE[x:f(x) \in M_j^r]$ which are not points of $k\lambda$ approach by $SE[x:f(x) \in M_j^r]$ is nowhere dense. Thus $E_r = \sum_j E_j$ is exhaustible. We need only to show that $E \subseteq E_r$. Choose ξ in E . Since ξ is not a point of $k\lambda$ f -approach by S , there exists an $M_1(f(\xi))$ such that for every $N(\xi)$, there exists an N contained in $N(\xi)$ such that $NSE[x:f(x) \in M_1(f(\xi))]$ does not have property λ . Choose $M_k^r(f(\xi))$ from $\{M^r\}$ such that $M_k^r(f(\xi)) \subseteq M_1(f(\xi))$. Since λ is ascending, $NSE[x:f(x) \in M_k^r(f(\xi))]$ does not have property λ . Hence $\xi \in E_r$. Thus E is exhaustible.

COROLLARY 1. The set of points of inexhaustible f -approach by X which are not points of concentrated inexhaustible f -approach by X is an exhaustible set [3].

Let ρ metrize Y and let $M(f(\xi); \epsilon)$ denote the set $E[y:\rho(y, f(\xi)) < \epsilon]$.

THEOREM I. Let λ be an ascending set property and S be a set in X . If the sequence of functions $\{f_n(x)\}$ is defined on X and converges uniformly to $F(x)$, and if there exists a point ξ in S such that for every n , ξ is a point of λ f_n -approach by S , then ξ is a point of λ F -approach by S .

PROOF. Let $M(F(\xi))$ and $N(\xi)$ be given. We must show that $N(\xi)SE[x:F(x) \in M(F(\xi))]$ has property λ . Choose a positive number ϵ such that $M(F(\xi); \epsilon) \subseteq M(F(\xi))$ and choose n such that for every x in S , $\rho(f_n(x); F(x)) < \epsilon/3$. Now $N(\xi)SE[x:f_n(x) \in M(\xi; \epsilon/3)]$ has property λ . Suppose $\eta \in N(\xi)SE[x:f_n(x) \in M(\xi; \epsilon/3)]$. Then $\rho(f_n(\eta); f_n(\xi)) < \epsilon/3$. Hence $\rho(F(\xi); F(\eta)) \leq \rho(F(\xi); f_n(\xi)) + \rho(f_n(\xi); f_n(\eta)) + \rho(f_n(\eta); F(\eta)) < \epsilon$. Thus $\eta \in N(\xi)SE$

$[x:F(x) \in M(F(\xi))]$. Since λ is an ascending set property, $N(\xi)SE[x:F(x) \in M(F(\xi))]$ has property λ .

COROLLARY II. *Let λ be the property that a set contains ξ as an interior point. Then Theorem I yields the well-known result that the uniform limit of a sequence of functions which are continuous at ξ is a function which is continuous at ξ .*

For other particular results, λ may be taken to be the property that a set contain a neighborhood, that a set be infinite, or any other ascending set property.

Let the complement with respect to X of a set S be denoted by S^{\sim} and the closure of S be designated cS .

DEFINITION IV. The property λ is said to be a *locally characterizing set property* if for every set S which has property λ and for every N such that NS has property λ , there exists an N_1 contained in N such that N_1S has property λ and $N_1(S^{\sim})$ does not have property λ .

For example, the property that a set be residual on a neighborhood is a locally characterizing set property while the property that a set be inexhaustible is not.

LEMMA III. *Let λ be an ascending locally characterizing set property. Let N be a neighborhood and R and S be sets in X such that NRS has property λ . Then there exists an N_1 contained in N such that $N_1R(S^{\sim})$ does not have property λ .*

PROOF. NS has property λ because λ is ascending. But since λ is locally characterizing, there exists an N_1 contained in N such that $N_1(S^{\sim})$ does not have property λ . From the ascending property of λ , it now follows that $N_1R(S^{\sim})$ does not have property λ .

LEMMA IV. *Let λ be an ascending locally characterizing set property. Let f be a function on X onto a subset of Y and let each element x of a set S in X be a point of λ f -approach by S . If ξ is a point of λ f -approach by S , then for every $M(f(\xi))$ and for every $N(\xi)$, there exists an N contained in $N(\xi)$ such that for every x in NS , $f(x) \in M(f(\xi))$.*

PROOF. Let $M(f(\xi))$ and $N(\xi)$ be given. Choose $M_1(f(\xi))$ such that $cM_1(f(\xi)) \in M(f(\xi))$. Now $N(\xi)SE[x:f(x) \in M_1(f(\xi))]$ has property λ . By Lemma III, there exists an N contained in $N(\xi)$ such that $NSE[x:f(x) \text{ not } \in M_1(f(\xi))]$ does not have property λ . Suppose η , a point of NS , is such that $f(\eta) \text{ not } \in M(f(\xi))$. Choose $M(f(\eta))$ disjoint from $M_1(f(\xi))$. Now $NSE[x:f(x) \in M(f(\eta))]$ has property λ . Since λ is ascending $NSE[x:f(x) \text{ not } \in M_1(f(\xi))]$ has property λ , a contradiction. Hence $f(\eta) \in M(f(\xi))$.

COROLLARY III. *Since ρ metrizes the space Y , under the conditions of Lemma IV, for every positive number ϵ , and for every $N(\xi)$, there exists an N contained in $N(\xi)$ such that for every x in NS , $\rho(f(x), f(\xi)) < \epsilon$.*

THEOREM II. *Let λ be an ascending locally characterizing set property. Let $f(x)$ be defined on X and let each element x of a set S in X be a point of λ f -approach by S . If ξ is a point of $k\lambda$ f -approach by S , then $f(x)$ is continuous at ξ with respect to S .*

PROOF. Let $M(f(\xi))$ and $N(\xi)$ be given. Choose $M_1(f(\xi))$ such that $\epsilon M_1(f(\xi)) \in M(f(\xi))$. Since ξ is a point of $k\lambda$ f -approach by S , there exists an $N_1(\xi)$ in $N(\xi)$ such that for every N in $N_1(\xi)$, $NS[x: f(x) \in M_1(f(\xi))]$ has property λ . Suppose η , a point of $N_1(\xi)S$, is such that $f(\eta) \notin M(f(\xi))$. Choose $M(f(\eta))$ disjoint from $M_1(f(\xi))$. Now $N_1(\xi)SE[x: f(x) \in M(f(\eta))]$ has property λ . By Lemma III, there exists an N_1 in $N_1(\xi)$, such that $N_1SE[x: f(x) \notin M(f(\eta))]$ does not have property λ . Since λ is ascending, $N_1SE[x: f(x) \in M_1(f(\xi))]$ does not have property λ , a contradiction. Thus $f(\eta) \in M(f(\xi))$. Hence $f(x)$ is continuous at ξ with respect to S .

Using Lemma II and Theorem II, we may state the following theorem.

THEOREM II'. *If λ is an ascending locally characterizing set property, if $f(x)$ is defined on X , and if each element x of S is a point of λ f -approach by S , then $f(x)$ is continuous over S with respect to S except on an exhaustible set.*

THEOREM III. *Let the sequence of functions $\{f_n(x)\}$ be defined and converge to $F(x)$ on X . Let λ be an ascending locally characterizing set property. Let R be a set residual in X and let it be required that for every n , each x in R is a point of λ f_n -approach by R . Then $F(x)$ is continuous over R with respect to R except on an exhaustible set.*

REMARK. If we say that a set has property λ if it contains a neighborhood and let $R = X$, we obtain Bledsoe's results for neighborhoodly functions [2]. The proof of Theorem III is similar to the proof given by Bledsoe, but we include it here because our theorem is more general.

PROOF. By Theorem II', $f_n(x)$ is continuous on a residual set R_n with respect to R . Thus for every n , $f_n(x)$ is continuous on the residual set $R' = \Pi_n R_n$ with respect to R . Let $w(x) = \limsup \eta \rightarrow x \rho(F(\eta), F(x))$ for η 's in R if this limit exists, and let $w(x) = 1$ at all other points of R . Let n and p be positive integers and

let $E_p^n = E[x \in R' : w(x) \geq 1/p \text{ and for every } m \geq n, \rho(f_m(x), F(x)) < 1/7p]$. Suppose there are integers n and p and a neighborhood N such that E_p^n is dense on N . Choose ξ in $E_p^n N$. Then $\rho(f_n(\xi), F(\xi)) < 1/7p$. Since $f_n(x)$ is continuous at ξ with respect to R , there exists an $N(\xi)$ contained in N such that for every x in $N(\xi)R$, $\rho(f_n(x), f_n(\xi)) < 1/7p$. Since $w(\xi) \geq 1/p$, there exists an x_1 in $N(\xi)R$ such that $\rho(F(\xi), F(x_1)) \geq 6/7p$. Choose $m \geq n$ such that $\rho(f_m(x_1), F(x_1)) < 1/7p$. Since x_1 is a point of λ f_m -approach by R , using Corollary III, we can conclude that there exists an N_1 contained in $N(\xi)$ such that for every x in N_1R , $\rho(f_m(x), f_m(x_1)) < 1/7p$. In particular let x_2 be contained in $N_1E_p^n$. Then $\rho(f_m(x_2), f_m(x_1)) < 1/7p$. But $\rho(f_m(x_2), f_n(x_2)) < 2/7p$. We have $\rho(F(\xi), F(x_1)) \leq \rho(F(\xi), f_n(\xi)) + \rho(f_n(\xi), F(x_1)) + \rho(f_n(\xi), f_n(x_2)) + \rho(f_n(x_2), f_m(x_2)) + \rho(f_m(x_2), f_m(x_1)) < 6/7p$, a contradiction. Thus $E = \sum_{n,p} E_p^n$ is exhaustible and $F(x)$ is continuous on $(R' - E)$ with respect to R .

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